APPROXIMATING $\bar{z}$ IN THE BERGMAN SPACE

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Abstract. We consider the problem of finding the best approximation to $\bar{z}$ in the Bergman Space $A^2(\Omega)$. We show that this best approximation is the derivative of the solution to the Dirichlet problem on $\partial\Omega$ with data $|z|^2$ and give examples of domains where the best approximation is a polynomial, or a rational function. Finally, we obtain the "isoperimetric sandwich" for $dist(\bar{z}, \Omega)$ that yields the celebrated St. Venant inequality for torsional rigidity.

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}$ with boundary Γ. Recall that the Bergman space $A^2(\Omega)$ is defined by:

$$A^2(\Omega) := \{f \in Hol(\Omega) : \|f\|^2_{A^2(\Omega)} = \int_\Omega |f(z)|^2 dA(z) < \infty\}.$$

In [10] the authors studied the question of "how far" $\bar{z}$ is from $A^2(\Omega)$ in the $L^2(\Omega)$-norm. They showed that the best approximation to $\bar{z}$ in this setting is 0 if and only if Ω is a disk, and that the best approximation is $\frac{c}{z}$ if and only if Ω is an annulus centered at the origin. In this note, we examine the question of what the best approximation looks like in other domains. In section 2, we characterize the best approximation to $\bar{z}$ as the derivative of the solution to the Dirichlet problem on Γ with data $|z|^2$. This shows an interesting connection between the Dirichlet problem and the Bergman projection. Recently in [14], A. Legg noted independently another such connection via the Khavinson-Shapiro conjecture. (Recall that the latter conjecture states that ellipsoids are the only domains where the solution to the Dirichlet problem with polynomial data is always a polynomial, cf. [15] and [18]. In [14, Proposition 2.1], the author showed that in the plane this happens if and only if the Bergman projection maps polynomials to polynomials). In section 3 we look at specific examples. In particular we look at domains for which the best approximation is a monomial Cz^k, some examples where the best approximation is a rational function with simple poles, as well as examples where the best approximation is a rational function with non-simple poles. In section 4, we prove two isoperimetric inequalities, and obtain the St. Venant inequality.

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2. Results

The following theorem is the high ground for the problem.

Theorem 1. *Let Ω be a bounded finitely connected domain. Then $f(z)$ is the projection of $\overline{z}$ onto $A^2(\Omega)$ if and only if $|z|^2 = F(z) + \overline{F(z)}$ on $\Gamma = \partial\Omega$, where $F'(z) = f(z)$.*

(Although F can, in a multiply connected domain, be multivalued, $Re(F)$ can be assumed to be single valued as a solution to the Dirichlet problem with data $|z|^2$ on Γ.)

Proof. First suppose that $\overline{z} - f(z)$ is orthogonal to $A^2(\Omega)$ in $L^2(\Omega)$. Then for every $z \in \hat{\mathbb{C}}\backslash\overline{\Omega}$ we have that

$$\int_\Omega (\overline{\zeta} - f(\zeta))\frac{1}{\overline{\zeta - z}}dA(\zeta) = 0 = \int_\Omega (\zeta - \overline{f(\zeta)})\frac{1}{\zeta - z}dA(\zeta).$$

Then, by Green's Theorem, for any single valued branch of F, where $F' = f$, we have that

$$\int_\Gamma (|\zeta|^2 - \overline{F(\zeta)})\frac{1}{\zeta - z}d\zeta = 0.$$

By the F. and M. Riesz Theorem, this happens if and only if we have

$$|\zeta|^2 - \overline{F(\zeta)} = h(\zeta)$$

on Γ, where $h(\zeta)$ is analytic in Ω.

Now, since $|\zeta|^2$ is real and we have that $|\zeta|^2 = \overline{F(\zeta)} + h(\zeta)$ on Γ, then it must be that

$$\overline{F(\zeta)} + h(\zeta) = F(\zeta) + \overline{h(\zeta)},$$

which implies that $h = F$.

Conversely, if $|\zeta|^2 - \overline{F(\zeta)} = h(\zeta)$ on Γ for some $h(\zeta)$ analytic in Ω, then we have that for all $z \in \hat{\mathbb{C}}\backslash\overline{\Omega}$,

$$\begin{aligned} 0 &= \int_\Gamma (|\zeta|^2 - \overline{F(\zeta)})\frac{1}{\zeta - z}d\zeta \\ &= \int_\Omega (\zeta - \overline{F'(\zeta)})\frac{1}{\zeta - z}dA(\zeta), \end{aligned}$$

and so we have that $\overline{\zeta} - F'(\zeta)$ is orthogonal to $A^2(\Omega)$.

. □

This argument is similar to that of Khavinson and Stylianopoulos in [13]. The following is an immediate corollary.

Corollary 2. *The best approximation to $\overline{z}$ in $A^2(\Omega)$ is a polynomial if and only if the Dirichlet problem with data $|z|^2$ has a real-valued polynomial solution. Similarly, the best approximation to $\overline{z}$ in $A^2(\Omega)$ is a rational function if and only if the Dirichlet problem with data $|z|^2$ has a solution which is the sum of a rational function and a finite linear combination of logarithmic potentials of real point charges located in the complement of Ω.*

The following theorem, loosely speaking, shows that increasing the connectivity of the domain essentially improves the approximation.

Theorem 3. *Let Ω be a finitely connected domain and let $f(z)$ be the best approximation to $\overline{z}$ in $A^2(\Omega)$. Then f must have at least one singularity in every bounded component of the complement.*

Proof. Suppose $\partial\Omega = \Gamma = \cup_{i=1}^n \Gamma_i$ where Γ_i is a Jordan curve for each i. By Theorem 1, we must have that $|z|^2 - 2ReF = 0$ on Γ where $F' = f$. Suppose that there is a bounded component K of the complement of Ω such that f is analytic in $G := \Omega \cup K$. Without loss of generality we will assume $\partial G = \cup_{i=1}^{n-1} \Gamma_i$. Then $|z|^2 - 2ReF$ is subharmonic in G and vanishes on ∂G. However since $|z|^2 - 2ReF$ cannot be constant in G, it must be that $|z|^2 - 2ReF < 0$ in G. In particular it cannot vanish on Γ_n. □

The following noteworthy corollary is now immediate.

Corollary 4. *If Ω is a finitely connected domain, and the best approximation to $\overline{z}$ is a polynomial, then Ω must be simply connected and $\partial\Omega$ is algebraic.*

The converse to Corollary 4 is false. In Section 3, we will give an example of a simply connected domain where the best approximation to $\overline{z}$ is a rational function. Corollary 4 implies that if the best approximation to $\overline{z}$ is a polynomial then the boundary of Ω, $\partial\Omega$, possesses the Schwarz function (cf. [19]). There is a connection between the best approximation to $\overline{z}$ in $A^2(\Omega)$ and the Schwarz function of $\partial\Omega$. We record this connection in the following proposition.

Proposition 5. *If Ω is a simply connected domain, and if the best approximation to $\overline{z}$ is a polynomial of degree at least 1, then the Schwarz function of $\Gamma = \partial\Omega$ cannot be meromorphic in Ω. Further, when the best approximation is a polynomial the Schwarz function of the corresponding domain must have algebraic singularities and no finite poles unless Ω is a disk.*

Proof. Suppose that $S(z)$ is the Schwarz function of $\Gamma = \partial\Omega$ and $p(z)$, a polynomial of degree $n-1$, is the best approximation to $\overline{z}$ in $A^2(\Omega)$ with anti-derivative $P(z)$. By Theorem 1, $zS(z) = P(z) + \overline{P(z)} = P(z) + P^{\#}(S(z))$ on Γ, where $P^{\#}(z) = \overline{P(\overline{z})}$. If S has a pole of order k at some $z_0 \neq 0$, then $zS(z)$ has a pole of order k at z_0 while $P^{\#}(S(z))$ has a pole of order nk at z_0. Thus $n \leq 1$. If $z_0 = 0$, and $k \geq 2$, then the same argument applies. If $z_0 = 0$ and $k = 1$, then p is constant and Γ is a circle. Since S is meromorphic in Ω if and only if the conformal map $\varphi : \mathbb{D} \to \Omega$ is a rational function, this shows that if Ω is a quadrature domain which is not a disk, then the best approximation to $\overline{z}$ cannot be a polynomial (cf. [19, pp.17-19]). □

We now look at some examples illustrating the above results.

3. Examples

The following examples were generated using Maple by plotting the boundary curve $|z|^2 - 1 = Const\, \Re(F(z))$ where, by Theorem 1, $f(z) = \frac{F'(z)}{2}$ is the best approximation to $\overline{z}$ in $A^2(\Omega)$, and $\Re(F(z))$ is the real part of $F(z)$. Since F is unique up to a constant of integration, all such examples will be similar perturbations of a disk.

Note in the next few examples with best approximation Cz^k, the associated domains have the $k+1$ fold symmetry inherited from the k fold symmetry of the best approximation.

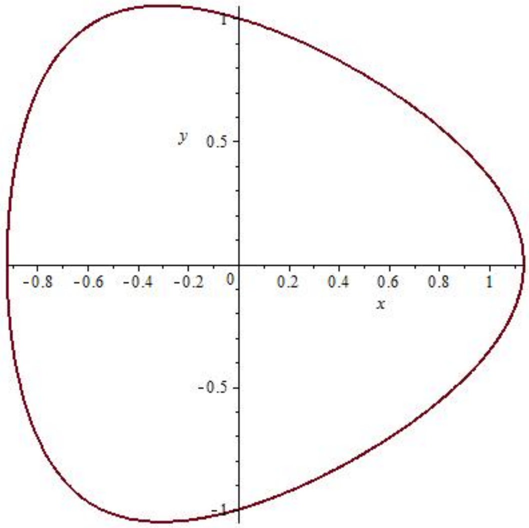

FIGURE 3.1

In Figure 3.1, the best approximation to $\overline{z}$ is $\frac{3z^2}{10}$.

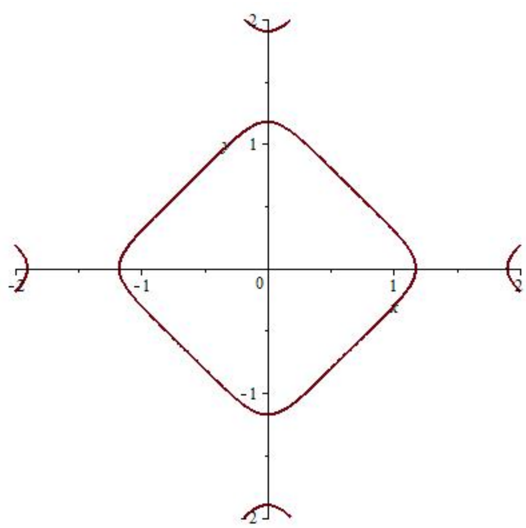

FIGURE 3.2

In Figure 3.2, the best approximation to $\overline{z}$ is $\frac{2z^3}{5}$.

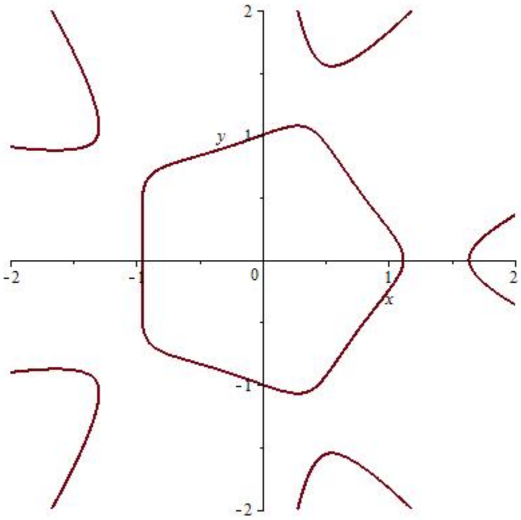

FIGURE 3.3

In Figure 3.3, the best approximation to $\overline{z}$ is $\frac{5z^4}{14}$.

The following example shows that the best approximation may be a rational function even when the domain is simply connected. Thus while Corollary 4 guarantees that Ω is simply connected whenever the best approximation to $\overline{z}$ is an entire function, the converse is not true.

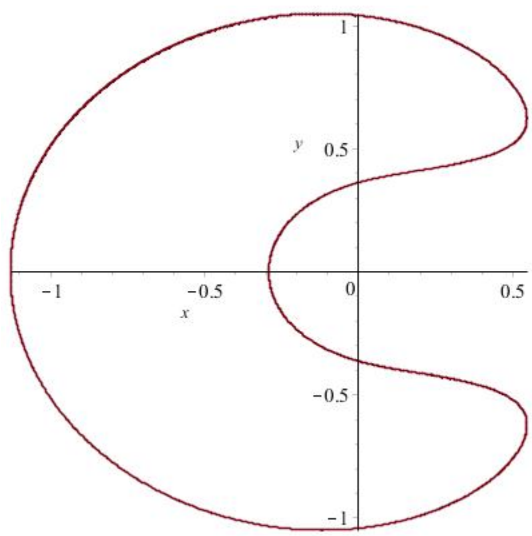

FIGURE 3.4

In this domain, the best approximation to $\overline{z}$ is $f(z) = \frac{1}{3z} + \frac{1}{5(z-\frac{1}{2})}$.

The constant(s) involved also play a strong role in the shape, and even connectivity of the domain, as the following pictures shows.

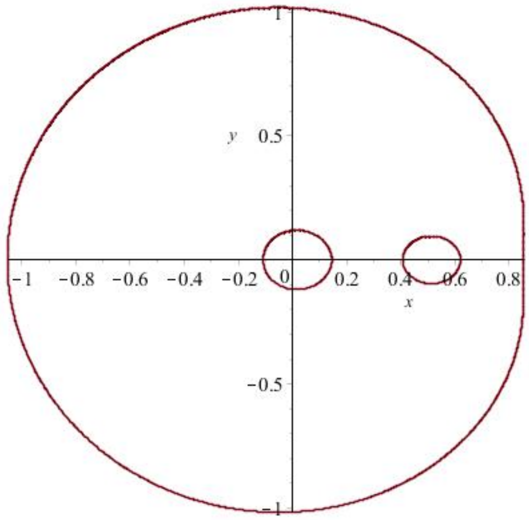

FIGURE 3.5

In Figure 3.5, the best approximation to $\overline{z}$ is $f(z) = \frac{1}{7z} + \frac{1}{10(z-\frac{1}{2})}$.

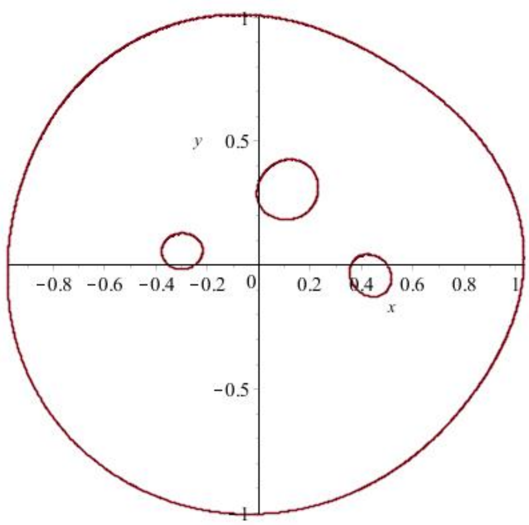

FIGURE 3.6

In Figure 3.6, the best approximation to $\overline{z}$ is $f(z) = -\frac{3z^2-2(\frac{1}{4}-\frac{1}{3}i)z-\frac{1}{8}+\frac{1}{12}i}{40(z-\frac{1}{2})^2(z-\frac{i}{3})^2(z+\frac{1}{4})^2}$.

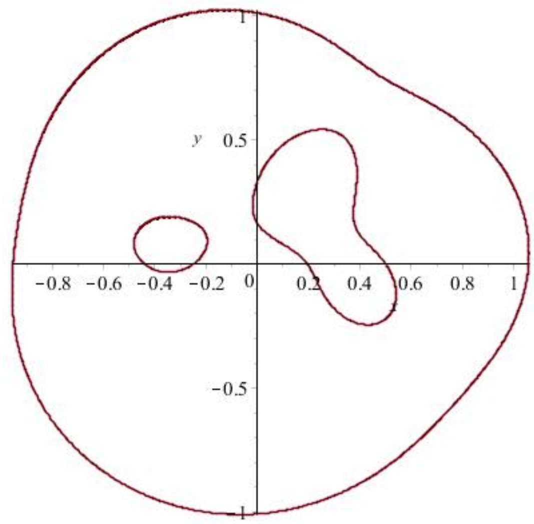

FIGURE 3.7

In Figure 3.7, the best approximation to $\overline{z}$ is $f(z) = -\frac{3z^2-2(\frac{1}{4}-\frac{1}{3}i)z-\frac{1}{8}+\frac{1}{12}i}{10(z-\frac{1}{2})^2(z-\frac{i}{3})^2(z+\frac{1}{4})^2}$.

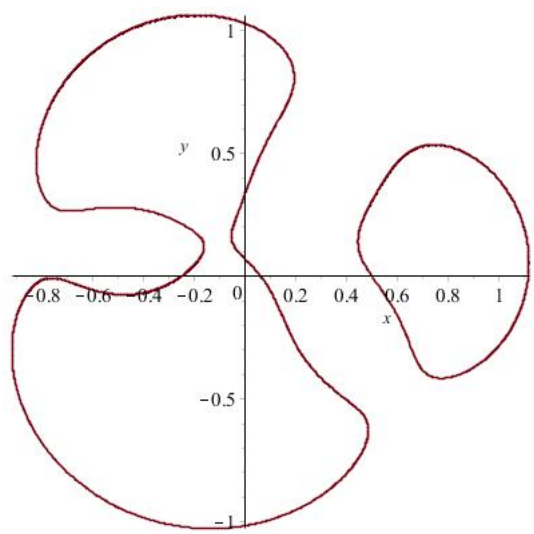

FIGURE 3.8

In Figure 3.8, the best approximation to $\overline{z}$ is $f(z) = -\frac{3z^2-2(\frac{1}{4}-\frac{1}{3}i)z-\frac{1}{8}+\frac{1}{12}i}{8(z-\frac{1}{2})^2(z-\frac{i}{3})^2(z+\frac{1}{4})^2}$. (It should be noted that in all of the above examples, the poles lie outside of $\overline{\Omega}$.)

As the order of the pole of the best approximation increases we see $k-1$ symmetric loops separating the pole from the domain. (Here k is the order of the pole of the best approximation) .

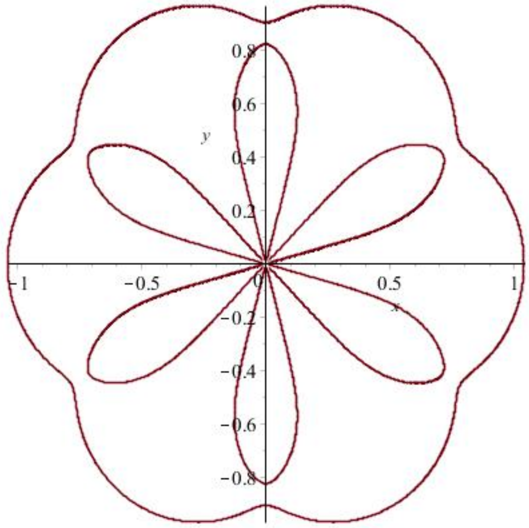

FIGURE 3.9

In Figure 3.9 the best approximation to $\overline{z}$ is $f(z) = \frac{-3}{10z^7}$. (It should be noted that the loops do not pass through 0. So 0 does not belong to $\overline{\Omega}$!)

4. Bergman Analytic Content

In [10] the authors expanded the notion of analytic content, $\lambda(\Omega) := \inf_{f\in H^\infty(\Omega)} \|\overline{z} - f\|_\infty$ defined in [6] and [11], to Bergman and Smirnov spaces context. The following "isoperimetric sandwich" goes back to [11]:

$$\frac{2A(\Omega)}{Per(\Omega)} \leq \lambda(\Omega) \leq \sqrt{\frac{A(\Omega)}{\pi}},$$

where $A(\Omega)$ is the area of Ω, and $Per(\Omega)$ is the perimeter of its boundary. Here the upper bound is due to Alexander (cf. [2]), and the lower bound is due to D. Khavinson (cf. [6] , [9], and[11]).

Following [10], we define $\lambda_{A^2}(\Omega) := \inf_{f\in A^2(\Omega)} \|\overline{z} - f\|_2$.

Theorem 6. *If Ω is a simply connected domain with a piecewise smooth boundary, then*

$$\sqrt{\rho(\Omega)} \leq \lambda_{A^2}(\Omega) \leq \frac{Area(\Omega)}{\sqrt{2\pi}},$$

where $\rho(\Omega)$ is the torsional rigidity of Ω (cf. [17, pg. 24]*).*

Proof. To see the lower bound, we note that by duality

$$\lambda_{A^2}(\Omega) := \inf_{f\in A^2(\Omega)} \|\overline{z} - f\|_2 = \sup_{g\in (A^2(\Omega))^\perp} \left| \frac{1}{\|g\|_2} \int_\Omega \overline{zg} dA(z) \right|. \tag{4.1}$$

By Khavin's lemma (cf. [5] ,[10] and [19]), we have that

$$(A^2(\Omega))^\perp := \{\frac{\partial u}{\partial z} |\, u \in W_0^{1,2}(\Omega)\},$$

where $W_0^{1,2}(\Omega)$ is the standard Sobolev space of functions with square-integrable gradients and vanishing boundary values. Thus, integrating by parts, (4.1) can be written as

$$\lambda_{A^2(\Omega)} = \sup_{u \in W_0^{1,2}(\Omega)} \frac{1}{\left\|\frac{\partial u}{\partial z}\right\|_2} \left| \int_\Omega u dA(z) \right|.$$

Any particular choice of $u(z)$ will thus yield a lower bound. Suppose we choose $u(z)$ to be the stress function satisfying

$$\begin{cases} \Delta u = -2 \\ u|_{\partial\Omega} = 0 \end{cases}$$

(cf. [5] and [17]). Then, since $u(z)$ is real-valued, we have that $\left\|\frac{\partial u}{\partial z}\right\|_2 = \frac{1}{2}\left\|\nabla u\right\|_2$ and

$$\frac{1}{\left\|\frac{\partial u}{\partial z}\right\|_2} \left| \int_\Omega u dA(z) \right| = \frac{2\left|\int_\Omega u dA(z)\right|}{\left\|\nabla u\right\|_{L^2(\Omega)}} = \sqrt{\rho(\Omega)},$$

(cf. [5] and [16]). Thus,

$$\lambda_{A^2}(\Omega) \geq \sqrt{\rho(\Omega)}. \tag{4.2}$$

To prove the upper bound, observe that

$$\lambda^2_{A^2}(\Omega) = \left\|\overline{z}\right\|^2 - \left\|P(\overline{z})\right\|^2,$$

where P is the Bergman projection. Let T_z be the Toeplitz operator acting on $A^2(\Omega)$ with symbol $\varphi(z) = z$, and let $[T_z^*, T_z] = T_z^* T_z - T_z T_z^*$ be the self-commutator of T_z. In [16], it was proved that

$$\left\|[T_z^*, T_z]\right\| = \sup_{g \in A_1^2(\Omega)} \left(\left\|\overline{z}g\right\|^2 - \left\|P(\overline{z}g)\right\|^2\right) \leq \frac{Area(\Omega)}{2\pi},$$

where $A_1^2(\Omega) = \{g \in A^2(\Omega) : \ \|g\|_2 = 1\}$. Taking $g = \frac{1}{\sqrt{Area(\Omega)}}$ yields

$$\frac{1}{Area(\Omega)}\left(\left\|\overline{z}\right\|^2 - \left\|P(\overline{z})\right\|^2\right) \leq \frac{Area(\Omega)}{2\pi},$$

and the upper bound follows. □

The celebrated St. Venant inequality (cf. [17]) follows immediately.

Corollary 7. *Let Ω be a simply connected domain. Then*

$$\rho(\Omega) \leq \frac{Area^2(\Omega)}{2\pi}.$$

5. Concluding Remarks

Recall that for all $u \in W_0^{1,2}(\Omega)$, we may write

$$\left| \int_\Omega u(z) dA(z) \right| = \left| \int_\Omega \frac{-1}{\pi} \int_\Omega \frac{\partial u}{\partial \overline{\zeta}} \frac{1}{\zeta - z} dA(\zeta) dA(z) \right|. \tag{5.1}$$

Applying Fubini's Theorem and the Cauchy-Schwartz inequality, we find that

$$\left| \int_\Omega u(z) dA(z) \right| \leq \left\| \frac{\partial u}{\partial \overline{z}} \right\|_2 \left\| \frac{1}{\pi} \int_\Omega \frac{dA(z)}{z - \zeta} \right\|_2. \tag{5.2}$$

In [7] and [8], (also cf. [3]) it was proved that the Cauchy integral operator $C : L^2(\Omega) \to L^2(\Omega)$, defined by

$$Cf(z) = \frac{-1}{\pi}\int_\Omega \frac{f(\zeta)}{\zeta - z} dA(\zeta),$$

has norm $\frac{2}{\sqrt{\Lambda_1}}$ whenever Ω is a simply connected domain with a piecewise smooth boundary, and Λ_1 is the smallest positive eigenvalue of the Dirichlet Laplacian,

$$\begin{cases} -\Delta u = \Lambda u \\ u|_{\partial\Omega} = 0 \end{cases}.$$

Further, by the Faber-Krahn inequality , cf. [17, pp. 18, 98] and [4, p. 104], we have that

$$\frac{2}{\sqrt{\Lambda_1}} \le \frac{2}{j_0}\sqrt{\frac{Area(\Omega)}{\pi}},$$

where j_0 is the smallest positive zero of the Bessel function $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}(\frac{x}{k})^{2k}$. Combining the above inequality with (5.2) we obtain

$$\frac{1}{\left\|\frac{\partial u}{\partial \overline{z}}\right\|_2}\left|\int_\Omega u dA(z)\right| \le \frac{2}{j_0}\frac{Area(\Omega)}{\sqrt{\pi}}. \tag{5.3}$$

This together with (4.2) and (5.2), yields an isoperimetric inequality:

$$\rho(\Omega) \le \frac{4Area^2(\Omega)}{j_0^2\pi}.$$

However, this is a coarser upper bound than that found above since $\frac{2}{j_0} \ge \frac{1}{\sqrt{2}}$. Since this upper bound depends entirely on $\left\|\frac{1}{\pi}\int_\Omega \frac{dA(z)}{z-\zeta}\right\|_2$, and since in the case when Ω is a disk D we find that $\left\|\frac{1}{\pi}\int_D \frac{dA(z)}{z-\zeta}\right\|_2 = \frac{Area(D)}{\sqrt{2\pi}}$, we conjecture, in the spirit of the Ahlfors-Beurling inequality (cf. [1] and [9]), that

$$\left\|\frac{1}{\pi}\int_\Omega \frac{dA(z)}{z-\zeta}\right\|_2 \le \frac{Area(\Omega)}{\sqrt{2\pi}}.$$

If true, this would provide an alternate proof to the upper bound for Bergman analytic content, as well as a more direct proof of the St. Venant inequality.

One is tempted to ask if any connection can be made between "nice" best approximations and the order of algebraic singularities of the Schwarz function. For example when Ω is an ellipse, the Schwarz function has square root singularities at the foci, and the best approximation to $\overline{z}$ is a linear function.

We would also like to find bounds on constants C which guarantee that the solution to the equation $|z|^2 - 1 = C(z^n + \overline{z}^n)$ is a curve which bounds a Jordan domain. This seems to depend on n.

It would also be interesting to examine similar questions for the Bergman space $A^p(\Omega)$ when $p \ne 2$, as well as similar questions for the best approximation of $|z|^2$ in $L^2_h(\Omega)$, the closed subspace of functions harmonic in Ω and square integrable with respect to area. However, it's not clear what the analog of Theorem 1 would be in this case. Mimicking the proof of Theorem 1 runs aground quickly.

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